Quantum Morphisms

Lecture 7

Last Week

$\alpha_p(G) \leq n^0(M) + \min\{n^+(M), n^-(M)\}$ for any weighted adjacency mtx M.

Inertia bound is not always tight:

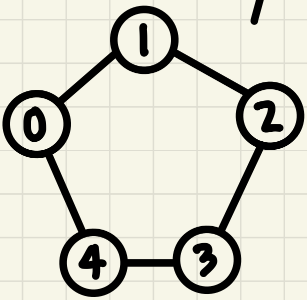
$$\alpha_p(L(Paley(9))) = 4.5$$

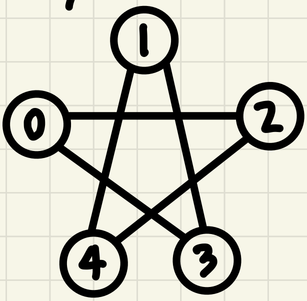
Exercises

1) Let C_n be the cycle of length n. Show that $C_n \xrightarrow{q} H \Leftrightarrow C_n \to H$ for any graph H.

2) Let G be a connected graph. Show that if $G \xrightarrow{q} H$ then there is a connected component H' of H such that $G \xrightarrow{q} H'$.

Graph Isomorphism





An isomorphism from a graph G to a graph H is a function $f: V(G) \to V(H)$ such that

1) f is a bijection,
2) $g \sim g' \Leftrightarrow f(g) \sim f(g')$.

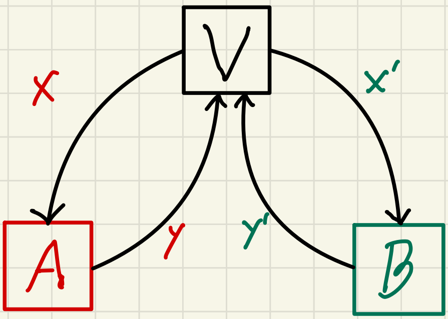
Example: $f(i) = 2i \bmod 5$ for the two graphs above.

Note: f is an isomorphism from G to H if and only if f^{-1} is an isomorphism from H to G

If an isomorphism from G to H exists, then we say that G and H are isomorphic and write $G \cong H$.

Quantum Isomorphism

(G,H)-Isomorphism Game



- Assume $V(G)$ + $V(H)$ are disjoint
- V sends A + B vertices $x, x' \in V(G) \cup V(H)$
- A + B respond with $y, y' \in V(G) \cup V(H)$
- As usual A + B cannot communicate during the game

Winning conditions

1) $x \in V(G) \Leftrightarrow y \in V(H)$, same for x', y'.

   Thus $\{x,y\} = \{g,h\}$ for some $g \in V(G)$, $h \in V(H)$.

   Define g', h' similarly.

2) $\mathrm{rel}(g,g') = \mathrm{rel}(h,h')$ i.e. $g = g' \Leftrightarrow h = h'$; $g \sim g' \Leftrightarrow h \sim h'$; $g \not\simeq g' \Leftrightarrow h \not\simeq h'$

Remark: The (G,H), $(\overline{G},\overline{H})$, and (H,G)-isomorphism games are all the same.

Proposition: There is a perfect classical strategy for the (G,H)-isomorphism game if and only if $G \cong H$.

$f: V(G) \to V(H)$ iso respond to $\overset{h}{g}$ with $\overset{f^{-1}(h)}{f(g)}$

Definition: We say that G & H are quantum isomorphic, denoted $G \cong_q H$, if there is a perfect quantum strategy for the (G,H)-isomorphism game.

Recall: A quantum strategy for the (G,H)-isomorphism game consists of (let $V = V(G) \cup V(H)$)

1) unit vector $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ for some $d \in \mathbb{N}$;
2) POVMs $\mathcal{E}_x = \{E_{xy} \in \mathbb{C}^{d \times d} : y \in V\}$ $\forall x \in V$ for Alice;
3) POVMs $\mathcal{F}_x = \{F_{xy} \in \mathbb{C}^{d \times d} : y \in V\}$ $\forall x \in V$ for Bob.

This produces the correlation

$$p(y,y'|x,x') = \langle \psi | E_{xy} \otimes F_{x'y'} | \psi \rangle.$$

To win they need $p(y,y'|x,x')=0$ unless there are $g,g'\in V(G)$ & $h,h'\in V(H)$ s.t. $\{x,y\}=\{g,h\}$, $\{x',y'\}=\{g',h'\}$ and $rel(g,g')=rel(h,h')$.

Theorem: If $G\cong_q H$ then there is a winning quantum strategy s.t.

1) $|\psi\rangle=\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle\in\mathbb{C}^d\otimes\mathbb{C}^d$ for some $d\in\mathbb{N}$;

2) E_{xy} & F_{xy} are projections $\forall x,y\in V(G)\cup V(H)$;

3) $F_{xy}=E_{xy}^T$ $\forall x,y\in V(G)\cup V(H)$.

Furthermore,

4) $E_{xy}=0$ if $x,y\in V(G)$ or $x,y\in V(H)$;

5) $E_{hg}=E_{gh}$ $\forall g\in V(G)$ & $h\in V(H)$.

this is analogous to saying that the strategy for inputs from $V(G)$ is "inverse" to the strategy for inputs from $V(H)$.

If A got g & responded w/ h, & B got h, then he must respond with g

Corollary: $G \cong_q H$ if and only if there exist $d \in \mathbb{N}$ and projections $E_{gh} \in \mathbb{C}^{d \times d}$ $\forall g \in V(G), h \in V(H)$ satisfying:

1) $\sum_h E_{gh} = I \quad \forall g \in V(G)$;
2) $\sum_g E_{gh} = I \quad \forall h \in V(H)$;
3) $E_{gh} E_{g'h'} = 0$ if $\mathrm{rel}(g,g') \neq \mathrm{rel}(h,h')$.

Remark: E_{gh} corresponds in some sense to mapping g to h, i.e. $E_{gh} = 0$ means g is <u>never</u> mapped to h and $E_{gh} = I$ means g is <u>always</u> mapped to h.

<u>Matrix Formulations</u>

Isomorphism:

$G \cong H \Leftrightarrow \exists$ a permutation matrix P s.t. $P^T A_G P = A_H$

$A_G P = P A_H$

Recall: $(A_G)_{uv} = \begin{cases} 1 & u \sim v \\ 0 & \text{o.w.} \end{cases}$

What is a permutation matrix?

$P \in \mathbb{C}^{n \times n}$ is a permutation matrix if

- $P_{ij} \in \{0,1\} \quad \forall\, i,j \in [n]$, and
- each row & column contains exactly one 1

Let's relax this notion.

Let $D \in \mathbb{C}^{n \times n}$. If

- D is real & $D_{ij} \in [0,1]$, and
- $\sum_j D_{ij} = 1 = \sum_\ell D_{\ell k} \quad \forall\, i,k \in [n]$,

then D is doubly stochastic.

Birkhoff-von Neumann Theorem: The set of doubly stochastic matrices is the convex hull of the permutation matrices.

From our relaxation of permutation matrices we obtain a relaxation of isomorphism:

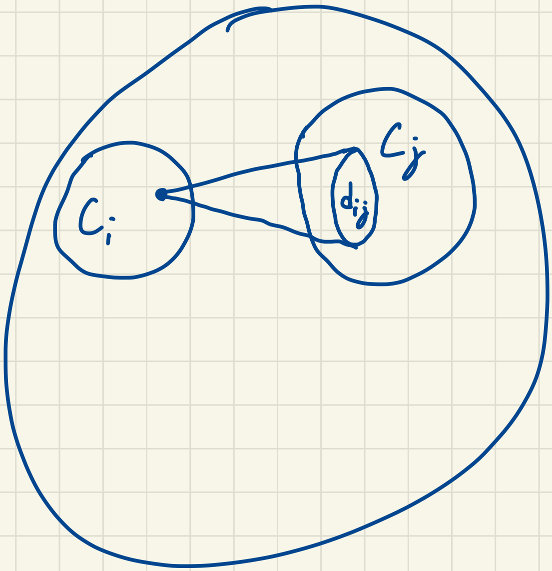
G and H are fractionally isomorphic, denoted by $G \cong_f H$, if there exists doubly stochastic D s.t.

$$A_G D = D A_H. \text{ - linear}$$

$$D^T A_G D = A_H \text{ gives isomorphism}$$

Theorem (Tinhofer/Ramana, Scheinerman, + Ullman)

$G \cong_f H$ if and only if G + H have equitable partitions $(C_1, \ldots, C_r)$ + $(C_1', \ldots, C_r')$ respectively s.t. $|C_i| = |C_i'|$ for all $i \in [r]$ and these partitions have the same quotient matrix.



Quotient Matrix = (d_{ij})

Algorithm for finding coarsest equitable partition:

1) Label vertices by degrees
2) Refine labelling by appending multiset of labels of neighbors
3) Repeat (2) until partition stabilizes.

Theorem (Atserias, Mančinska, Roberson, Šámal, Severini, Varvitsiotis)

$G \cong_f H \iff \exists$ a non-signalling correlation $p : (V(G) \cup V(H))^4 \to [0,1]$ that wins the (G,H)-isomorphism game.

Recall this means that

$\sum_{y'} p(y,y'|x,x')$ does not depend on x';

$\sum_{y} p(y,y'|x,x')$ does not depend on x, and

$p(y,y'|x,x') = 0$ on losing question/answer pairs.

Proof: Exercise.

Corollary: $G \cong_q H \Rightarrow G \cong_f H$.

Another relaxation:

Any permutation matrix P satisfies $P^T P = P P^T = I$, i.e. is unitary.

G & H are cospectral $\iff \exists$ unitary U s.t. $U^* A_G U = A_H$.

A "quantum" relaxation

A matrix $\mathcal{P} = (P_{ij}) \in M_n(\mathbb{C}^{d\times d}) = \mathbb{C}^{n\times n} \otimes \mathbb{C}^{d\times d}$ is a quantum permutation matrix (or magic unitary) if

$P_{ij} \in \mathbb{C}^{d\times d}$

1) $P_{ij} = P_{ij}^2 = P_{ij}^* \quad \forall\, i,j \in [n]$;
2) $\sum_j P_{ij} = I = \sum_\ell P_{\ell k} \quad \forall\, i,k \in [n]$.

$\Rightarrow$ $P_{ij}P_{ik} = 0$ if $j \neq k$, $P_{ij}P_{\ell j} = 0$ if $i \neq \ell$

Lemma: Suppose $\mathcal{P} = (P_{ij}) \in M_n(\mathbb{C}^{d\times d})$ s.t. $P_{ij} = P_{ij}^2 = P_{ij}^*\ \forall\, i,j$. Then $\mathcal{P}$ is unitary if and only if it is a quantum perm mtx.

Proof: Exercise.

Theorem (Atserias et al): $G \cong_q H \Leftrightarrow \exists$ a quantum permutation matrix $\mathcal{P} = (P_{gh}) \in M_n(\mathbb{C}^{d\times d})$ satisfying

$$\mathcal{P}^*(A_G \otimes I_d)\mathcal{P} = A_H \otimes I_d.$$

$$\Leftrightarrow \quad (A_G \otimes I_d)\mathcal{P} = \mathcal{P}(A_H \otimes I_d)$$

(g,h)-blocks: $\quad \sum_{g' \sim g} P_{g'h} = \sum_{h' \sim h} P_{gh'}$

Proof: Recall that $G \cong_q H \Leftrightarrow$ there are projections $P_{gh} \in \mathbb{C}^{d \times d}$ $\forall g \in V(G), h \in V(H)$ satisfying:

$$\left.\begin{array}{l} 1)\ \sum_h P_{gh} = I \quad \forall g \in V(G); \\ 2)\ \sum_g P_{gh} = I \quad \forall h \in V(H); \end{array}\right\} \mathcal{P} = (P_{gh}) \text{ is a quantum permutation matrix}$$

3) $P_{gh} P_{g'h'} = 0$ if $rel(g,g') \neq rel(h,h')$.

So it suffices to show that for a quantum permutation matrix $\mathcal{P} = (P_{gh})$, Condition (3) is equivalent to $(A_G \otimes I)\mathcal{P} = \mathcal{P}(A_H \otimes I)$

i.e. $\sum_{g' \sim g} P_{g'h} = \sum_{h' \sim h} P_{gh'}$ $\forall g, h$.

Suppose (3) holds. Then for any $g \in V(G)$, $h \in V(H)$:

$$\sum_{g' \sim g} P_{g'h} = \sum_{g' \sim g} P_{g'h} \sum_{h'} P_{gh'} = \sum_{g' \sim g} P_{g'h} \sum_{h' \sim h} P_{gh'} = \sum_{g'} P_{g'h} \sum_{h' \sim h} P_{gh'} = \sum_{h' \sim h} P_{gh'}.$$

Conversely, suppose that $\sum_{g' \sim g} P_{g'h} = \sum_{h' \sim h} P_{gh'}$ $\forall g, h$.

Note that these sums are projections. Thus

$$\sum_{g' \sim g} P_{g'h} \sum_{h' \sim h} P_{gh'} = \Big(\sum_{g' \sim g} P_{g'h}\Big)^2 = \sum_{g' \sim g} P_{g'h} = \sum_{g' \sim g} P_{g'h} \sum_{h'} P_{gh'}.$$

Therefore,

$\sum_{g'\sim g} P_{g'h} \sum_{h'\not\sim h} P_{gh'} = 0 \Rightarrow$ if $\hat{g}\sim g$ & $\hat{h}\not\sim h$, then

$$P_{\hat{g}h} P_{g\hat{h}} = P_{\hat{g}h}\Big(\sum_{g'\sim g} P_{g'h} \sum_{h'\not\sim h} P_{gh'}\Big) P_{g\hat{h}} = 0.$$

Similar for other cases of $\mathrm{rel}(g,\hat{g}) \neq \mathrm{rel}(h,\hat{h})$. $\square$

Corollary: If $G \cong_q H$, then G & H are cospectral.

$G \cong_q H \Rightarrow G \xrightarrow{q} H$ & $H \xrightarrow{q} G$ & $\overline{G} \xrightarrow{q} \overline{H}$ & $\overline{H} \xrightarrow{q} \overline{G}$

$\vartheta(G) = \vartheta(H)$ $\quad \bar{\vartheta}(G) = \bar{\vartheta}(H)$

$\chi_q, \alpha_p, \omega_q, \xi_f$